THE APPLICATION OF THE THREE-DIMENSIONAL THEORY OF ELASTICITY TO THE ANALYSIS OF FLEXURAL WAVES IN A SEMI-INFINITE PLATE ACTED ON BY A SHORT-TIME BOUNDARY LOADING

(PRIMENENIE TREKHMERNOI TEORII UPRUGOSTI K ANALIZU VOLNOVOGO PROTSESSA IZGIBA POLUBESKONECHNOI PLITY PRI KRATKOVREMENNO DEISTVUIUSHCHEI KRAEVOI NAGRUZKE)

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The wave propagation of flexural-type deformations, excited by the application of short-time loads on infinite or semi-infinite plates and beams, has been studied in the literature on the basis of the Kirchhoff theory [1-4] and a theory of the Timoshenko type [5-15]. The latter was first applied by Ufliand [5], who made use of the attractive integral-transform mathematical apparatus. Owing to its hyperbolic character, this theory eliminated essential defects [1,5] inherent in the parabolic Kirchhoff theory. Inaccuracies which occurred in the first papers when integral transform methods were applied were quickly corrected [5,6], and approximate methods for inverting the contour integrals that result from Timoshenko-type theories were worked out [8-11,14,15]. However, the problem of the accuracy of a Timoshenko-type theory in wave propagation problems remained almost unstudied.

Below, we apply the three-dimensional theory of elasticity. As a concrete example, we consider the waves excited in a semi-infinite plate by suddenly applied boundary stresses linearly distributed over the thickness of the plate. By means of a two-fold integral transform method, analogous to that applied in [16,17], a formal solution is established in the form of an infinite sum of one-dimensional integrals of the Laplace transform (Sections 1-6). We examine their approximate inversion by means of the saddle-point method and present numerical results for the first six integrals (Sections 7-9). We then study the error in the

Kirchhoff theory and in the Timoshenko-type theory (Sections 10-14). In carrying out this work, use was made of the experience gained in applying the three-dimensional theory to mixed problems [16-20].

1. We consider a state of stress that depends on a longitudinal coordinate x_1, a normal coordinate x_3 and the time t.

Let E be the modulus of elasticity, μ Poisson's ratio, $2h$ the thickness of the plate, c_1 and c_2 velocities of propagation of dilatational and shear (distortional) waves, ξ and ζ dimensionless coordinates, τ a dimensionless time, u and w dimensionless (divided by h) displacements in the x_1 and x_3 directions respectively, w_0 the normal displacement of the mean surface $\zeta = 0$, and σ_{ij} $(i,\ j = 1,\ 3)$ stresses multiplied by $(1 + \mu)E^{-1}$. Here

$$\xi = \frac{x_1}{h}, \qquad \zeta = \frac{x_3}{h}, \qquad \tau = \frac{tc_2}{h}, \qquad k^2 = \frac{c_2^2}{c_1^2} = \frac{1-2\mu}{2-2\mu} \tag{1.1}$$

We define such quantities as the bending moment, the transverse force, and the integrated displacement by means of the formulas

$$M = \int_{-1}^{+1} \sigma_{11}\zeta\, d\zeta, \qquad Q = \int_{-1}^{+1} \sigma_{13}\, d\zeta \tag{1.2}$$

$$U = \frac{3}{2}\int_{-1}^{+1} u\zeta\, d\zeta, \qquad W = \frac{1}{2}\int_{-1}^{+1} w\, d\zeta \tag{1.3}$$

For differentiation symbols we introduce the abbreviated notations

$$\partial_1 = \frac{\partial}{\partial\xi}, \qquad \partial_3 = \frac{\partial}{\partial\zeta}, \qquad \partial_\tau = \frac{\partial}{\partial\tau} \tag{1.4}$$

In this notation, the equations of equilibrium in terms of the displacements have the form

$$\begin{aligned} &\partial_3{}^2 u + k^{-2}\partial_1{}^2 u + (1-2\mu)^{-1}\partial_1\partial_3 w = \partial_\tau{}^2 u \\ &\partial_3{}^2 w + k^2\partial_1{}^2 w + (2-2\mu)^{-1}\partial_1\partial_3 u = k^2\partial_\tau{}^2 w \end{aligned} \tag{1.5}$$

On the basis of well-known relations in the theory of elasticity, the following equations for the integral quantities (1.2) and (1.3) result from (1.5):

$$\partial_1 M - Q = {}^1/_3\, \partial_\tau{}^2 U, \qquad \partial_1 Q = \partial_\tau{}^2 W \tag{1.6}$$

2. To be specific, we take the following conditions for the problem at hand:

a) On the plane $\xi = 0$ of the semi-infinite plate

$$\sigma_{11}(0, \zeta; \tau) = \frac{\zeta}{1-\mu} A(\tau), \qquad w(0, \zeta; \tau) = 0 \tag{2.1}$$

b) On the planes $\zeta = \pm 1$

$$\sigma_{13}(0, \pm 1; \tau) = \sigma_{33}(0, \pm 1; \tau) = 0 \tag{2.2}$$

c) Zero initial conditions, i.e. $A(\tau)$ and all of the unknown quantities to be identically zero for $\tau < 0$.

From (2.1) it follows that

$$\partial_1 u(0, \zeta; \tau) = \frac{1-2\mu}{1-\mu} \sigma_{11}(0, \zeta; \tau) = \frac{2k^2\zeta}{1-\mu} A(\tau) \tag{2.3}$$

$$M(0; \tau) = \frac{2}{3-3\mu} A(\tau), \qquad w_0(0; \tau) = 0, \qquad W(0; \tau) = 0 \tag{2.4}$$

The first and second of conditions (2.4) may be used as boundary conditions in an approximate solution based on the Kirchhoff theory [1, p.409], and the first and third in an approximate solution based on a Timoshenko-type theory [9, 14].

3. We define the Laplace transform by means of the formulas

$$\int_0^\infty F e^{-s\tau} d\tau = F^L, \qquad F = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F^L e^{s\tau} ds \tag{3.1}$$

where F denotes an arbitrary quantity under consideration. In particular

$$\int_0^\infty A(\tau) e^{-s\tau} d\tau = A^L \tag{3.2}$$

The Fourier cosine and sine transforms are defined as

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F_1 \cos \varkappa\xi \, d\xi = F_1^c, \qquad F_1 = \sqrt{\frac{2}{\pi}} \int_0^\infty F_1^c \cos \varkappa\xi \, d\varkappa \tag{3.3}$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F_2 \sin \varkappa\xi \, d\xi = F_2^s, \qquad F_2 = \sqrt{\frac{2}{\pi}} \int_0^\infty F_2^s \sin \varkappa\xi \, d\varkappa \tag{3.4}$$

Following [16, 17], we multiply the first and second of equations (1.5) respectively by the quantities

$$\sqrt{\frac{2}{\pi}} e^{-s\tau} \cos \varkappa\xi, \qquad \sqrt{\frac{2}{\pi}} e^{-s\tau} \sin \varkappa\xi \tag{3.5}$$

and integrate from 0 to ∞ over both ξ and τ. Integrating by parts and taking (2.1) and (2.3) into account, we have

$$\partial_3^2 u^{Lc} + (1-2\mu)^{-1}\varkappa\partial_3 w^{Ls} - k^{-2}\varkappa^2 u^{Lc} - s^2 u^{Lc} = 2^{3/2}(1-\mu)^{-1}\pi^{-1/2}A^L\zeta$$
$$\partial_3^2 w^{Ls} - (2-2\mu)^{-1}\varkappa\partial_3 u^{Lc} - k^2\varkappa^2 w^{Ls} - s^2k^2w^{Ls} = 0 \qquad (3.6)$$

In the transform space, (3.6) is a system of ordinary differential equations.

Carrying out a similar calculation, which also proceeds from well-known relations in the theory of elasticity, we obtain formulas that relate transforms of stresses to transforms of displacements. On the basis of (2.2), the expressions

$$\sigma_{13}^{Lc} = \frac{1}{2}[\partial_3 u^{Lc} + \varkappa w^{Ls}], \qquad \sigma_{33}^{Ls} = \frac{1}{1-2\mu}[(1-\mu)\partial_3 w^{Ls} - \mu\varkappa u^{Lc}] \quad (3.7)$$

give boundary conditions for system (3.6). It is not difficult to construct the solution of system (3.6) in the form of the sum of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. Here one may use ready-made formulas for the general solution given (for some more general cases) in [21]. Omitting intermediate calculations, we give the formulas for the transforms of the displacements

$$u^{Lc} = \sqrt{\frac{2}{\pi}}\frac{A^L}{(1-\mu)^2\alpha^2\Phi}[u^\circ + (1-2\mu)\Phi\zeta]$$
$$w^{Ls} = \sqrt{\frac{2}{\pi}}\frac{i\lambda A^L}{(1-\mu)^2\alpha^2\beta^2\Phi}(w^\circ - \Phi) \qquad (3.8)$$

Here

$$\alpha^2 = \lambda^2 - k^2s^2, \quad \beta^2 = \lambda^2 - s^2, \quad \lambda = i\varkappa$$
$$\gamma^2 = \lambda^2 - \frac{1}{2}s^2, \quad \vartheta^2 = \frac{1-2\mu}{2}s^2 + \mu\lambda^2 \qquad (3.9)$$

$$\Phi = -\lambda^2\beta^2\cos\alpha\frac{\sin\beta}{\beta} + \gamma^4\cos\beta\frac{\sin\alpha}{\alpha} \qquad (3.10)$$

$$u^\circ = \vartheta^2\left(-\lambda^2\frac{\sin\beta}{\beta}\frac{\sin\alpha\zeta}{\alpha} + \gamma^2\frac{\sin\alpha}{\alpha}\frac{\sin\beta\zeta}{\beta}\right) + $$
$$+ \mu\lambda^2\left(\gamma^2\cos\beta\frac{\sin\alpha\zeta}{\alpha} - \beta^2\cos\alpha\frac{\sin\beta\zeta}{\beta}\right) \qquad (3.11)$$
$$w^\circ = \vartheta^2\left(-\beta^2\frac{\sin\beta}{\beta}\cos\alpha\zeta + \gamma^2\frac{\sin\alpha}{\alpha}\cos\beta\zeta\right) +$$
$$+ \mu\beta^2(\gamma^2\cos\beta\cos\alpha\zeta - \lambda^2\cos\alpha\cos\beta\zeta)$$

If s and A^L are considered as given quantities, then it is not hard to prove that the functions $u^{Lc}(\lambda)$ and $w^{Ls}(\lambda)$ will be single-valued functions in the complex plane λ. The points $\Phi = 0$ are their poles, while the points $\alpha = 0$ and $\beta = 0$ are not poles.

4. The remaining problem consists in computing the inverse of transforms (3.8). We first invert the sine and cosine transforms with respect to the coordinate ξ, and then the Laplace transform with respect to τ. This sequence of inversions was applied in [16,17,20], and the reverse in [18,19]. It has the following advantages: (a) the sine and cosine transforms can be inverted by means of the residue theorem and, (b) the first inversion can be carried out without concrete specification of the function $A(\tau)$.

Noting that u^{Lc} is an even and w^{Ls} is an odd function of λ, we rewrite the inversion formulas (3.3) and (3.4) for the displacements in the form

$$u^L = -\frac{i}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} u^{Lc}e^{\lambda\xi}d\lambda, \qquad w^L = -\frac{1}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} w^{Ls}e^{\lambda\xi}d\lambda \tag{4.1}$$

In the calculation of the integrals (4.1) we shall formally consider s and A^L as given quantities.

Having in mind the study of the outgoing (damped as $\xi \to \infty$) waves, it is necessary to consider the poles that have Re $s > 0$ and are in the half-plane Re $\lambda \leqslant 0$. Taking into account the properties of u^{Lc} and w^{Ls} indicated previously, we have, in accordance with the residue theorem

$$u^L = A^L \sum_{j=1}^{\infty} u_j{}^* e^{\lambda_j\xi}, \qquad w^L = A^L \sum_{j=1}^{\infty} w_j{}^* e^{\lambda_j\xi} \tag{4.2}$$

$$u_j{}^* = \frac{2}{(1-\mu)^2 s^2}\left[\frac{\lambda\beta^2 u^\circ}{\Phi_*}\right]_{\lambda=\lambda_j} \qquad w_j{}^* = \frac{2}{(1-\mu)^2 s^2}\left[\frac{\lambda^2 w^\circ}{\Phi_*}\right]_{\lambda=\lambda_j} \qquad (j = 1, 2, 3, \ldots) \tag{4.3}$$

where λ_j are the roots of the Rayleigh-Lamb [22,23] equation

$$\Phi = 0 \tag{4.4}$$

satisfying the condition Re $\lambda_j \leqslant 0$ for Re $s > 0$. The function Φ_* is taken in the following sense:

$$\frac{\partial\Phi}{\partial\lambda} = \frac{s^2}{\lambda\alpha^2\beta^2}\Phi_*, \qquad \Phi_* = \Phi_0 - \Phi_1 \tag{4.5}$$

where by considering (4.4) the functions Φ_0 and Φ_1 may be written in the form

$$\Phi_0 = \gamma^2 \left(\frac{\lambda^4}{2-2\mu} - \frac{5-4\mu}{4-4\mu} \lambda^2 s^2 + k^2 s^4\right) \frac{\sin\alpha}{\alpha} \cos\beta$$

$$\Phi_1 = \frac{1}{4} \lambda^2 s^2 \alpha^2 \beta^2 \frac{\sin\alpha}{\alpha} \frac{\sin\beta}{\beta} + \left(\frac{\lambda^2}{2-2\mu} - \frac{s^2}{4}\right) \lambda^2 \beta^2 \cos\alpha \cos\beta \tag{4.6}$$

5. On the basis of (4.2), we rewrite the inversion formula (3.1) in the form

$$F = \frac{1}{2\pi i} \sum_{j=1}^{\infty} \int_{a_j - i\infty}^{a_j + i\infty} A^L F_j^* e^{\tau\varphi_j} ds, \qquad \varphi_j = s + \frac{\xi}{\tau} \lambda_j \tag{5.1}$$

To compute $F = w$, we use $F_j^* = w_j^*$ in (5.1). Other quantities are dealt with in a similar way. We give the expanded formulas for quantities that are of special interest in the analysis of the accuracy of two-dimensional theories and that lend themselves to calculation at subsequent points

$$w_{0j}^* = -\frac{2}{(1-\mu)^2 s^2} \left\{\frac{\lambda^2}{\Phi_*} \left[\vartheta^2 \left(-\beta^2 \frac{\sin\beta}{\beta} + \gamma^2 \frac{\sin\alpha}{\alpha}\right) + \mu\beta^2 (\gamma^2 \cos\beta - \lambda^2 \cos\alpha)\right]\right\}_{\lambda=\lambda_j}$$

$$W_j^* = \frac{1}{(1-\mu)^2} \left\{\frac{\lambda^2}{\Phi_*} \frac{\sin\alpha}{\alpha} \left(\vartheta^2 \frac{\sin\beta}{\beta} - \mu\gamma^2 \cos\beta\right)\right\}_{\lambda=\lambda_j} \tag{5.2}$$

$$U_j^* = \frac{3}{(1-\mu)^3} \left\{\frac{\lambda}{\alpha^2 \Phi_*} \left[\vartheta^4 \frac{\sin\beta}{\beta} \frac{\sin\alpha}{\alpha} - 2\mu\vartheta^2\gamma^2 \frac{\sin\alpha}{\alpha} \cos\beta + \mu^2 \lambda^2 \beta^2 \cos\alpha \cos\beta\right]\right\}_{\lambda=\lambda_j}$$

$$M_j^* = \frac{s^2}{(1-\mu)^3} \left\{\frac{1}{\alpha^2 \Phi_*} \left[\lambda^2 \vartheta^2 \frac{\sin\beta}{\beta} \frac{\sin\alpha}{\alpha} - \mu\gamma^2 (\vartheta^2 - \lambda^2) \frac{\sin\alpha}{\alpha} \cos\beta + \mu^2 \lambda^2 \beta^2 \cos\alpha \cos\beta\right]\right\}_{\lambda=\lambda_j}$$

$$Q_j^* = s^2 \left\{\frac{W^*}{\lambda}\right\}_{\lambda=\lambda_j}$$

We note that on the basis of (4.4), terms of the form $\sin\beta \cos\alpha$ are excluded from expression (5.2). This significantly shortened the formulas and in the computations reduced the danger of error due to small differences of large numbers.

Hence, we have constructed a formal exact solution in the form of the sum of contour integrals. Termwise it satisfies all of the equations and relationships in the three-dimensional theory of elasticity, and also the initial conditions and conditions (2.2). It satisfies boundary conditions (2.1) in terms of sums of the form

$$\sum_{j=1}^{\infty} w_j^* = 0, \qquad \sum_{j=1}^{\infty} \sigma_{11j}^* = \frac{\zeta}{1-\mu} \tag{5.3}$$

Thus, it follows that

$$\sum_{j=1}^{\infty} w_{0j}^* = \sum_{j=1}^{\infty} W_j^* = 0, \qquad \sum_{j=1}^{\infty} M_j^* = \frac{2}{3-3\mu} \tag{5.4}$$

It is not possible to invert exactly the contour integrals that have been obtained and to sum the infinite series involved. Therefore we must resort to approximate methods of inversion and must limit the computation to a finite number of terms of series (5.1). The solution to the inversion problem is further complicated by the fact that data on the behavior of the roots λ_j in the complex plane s are not contained in the literature. All of the existing information [24-26] pertains to the imaginary axis ($s = i\Omega$).

6. We examine some of the properties of the integrand functions. These can be seen without going into numerical data or into a description of the cuts necessary to guarantee single-valuedness of the roots $\lambda_j(s)$ in the s-plane.

1) For every λ_j it is possible to determine a positive number b_j such that on the imaginary axis $s = i\Omega$ [24, 25]

$$\begin{aligned} \lambda_j &= im_j \text{ where } -\infty < \Omega \leqslant b_j, \\ \lambda_j &= -im_j \text{ where } +\infty > \Omega \geqslant b_j, \end{aligned} \qquad \frac{d\lambda_j}{ds} = -\frac{1}{\psi_j} \tag{6.1}$$

where m_j and ψ_j are real positive functions of $|\Omega| \geqslant b_j$. Plots of the group velocity ψ_j for the m_j mode are given in [24-26]. Plots of $\psi_j(\Omega)$ ($j = 1, 2, \ldots, 5$) are also shown in Fig. 1, where Curve 1 is for the three-dimensional theory, Curve 2 is for the Timoshenko-type theory and Curve 3 is for the Kirchhoff theory. As a basis of enumerating the roots, we require that the numbers b_j should form an increasing sequence.

For $\mu = 0.3$, we have [24]

$$b_1 = 0, \quad b_2 = {}^1/_2\pi, \quad b_3 = 4.708$$
$$b_4 = \sqrt{3.5}\pi, \quad b_5 = {}^5/_2\pi, \ldots$$

2) All of the roots are purely imaginary only on the imaginary axis $s = i\Omega$. However, among the $\lambda_j (j \geqslant 3)$ there are roots that take on imaginary values not only on intervals (6.1), but also in a narrow zone of the lower frequencies Ω [24-26].

In these regions λ_j may be determined from formulas of the form of

(6.1), with m_j negative and ψ_j positive as before.*

3) For $s = i\Omega \to \pm i\infty$ we have

$$\lambda_1 \to -\psi_R s, \qquad \lambda_j \to -s, \qquad \psi_R = \frac{c_R}{c_2} \qquad (j \geqslant 2) \tag{6.2}$$

where c_R is the propagation velocity of Rayleigh surface waves.

4) In the neighborhood of the point $s = 0$ [24], we have the following expansion for the roots λ_1 and λ_2:

$$\lambda^4 = -\frac{3-3\mu}{2} s^2 + O(s^3) \tag{6.3}$$

whereas for $\lambda_j (j \geqslant 3)$ we have the expansion

$$\lambda_j = q_j \left[1 - \frac{1-\mu}{2q_j^2 \sin^2 q_j} s^2 + O(s^4)\right] \tag{6.4}$$

where the $q_j (j = 3, 4, \ldots)$ are known non-zero roots [27] of the characteristic equation of static St. Venant boundary effects.

Some further remarks concerning quantities like (5.2) may be of interest. If $s \to 0$, then for $j = 1, 2$ we have

$$w_{0j}^* \sim W_j^* \to -\frac{1}{2\lambda_j^2}, \qquad M_j^* \to \frac{1}{3-3\mu}, \qquad Q_j^* \to -\frac{s^2}{2\lambda_j^3} \tag{6.5}$$

whereas for $j \geqslant 3$ these same quantities approach zero. Hence, in summations (5.4), the first two terms dominate for small s. With increasing s, the relative values of terms $j \geqslant 3$ increase rapidly. For purposes of numerical illustration, some data for the quantities

$$B_{j1} = w_{0j}^* (is)^{-1}, \qquad B_{j2} = W_j^* (is)^{-1}, \qquad B_{j3} = M_j^* (is)^{-1}, \qquad B_{j4} = Q_j^* s^{-1} \tag{6.6}$$

are given in Tables 1 and 2.

7. We examine the case

$$A(\tau) = A_0 H(\tau), \quad A^L = A_0 s^{-1}, \quad A_0 = \text{const} \tag{7.1}$$

where $H(\tau)$ is the Heaviside unit function.

We take $\mu = 0.3$ and for $\tau \to \infty$ we apply the saddle-point method to obtain asymptotic approximations of the contour integrals (5.1) for the quantities

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* In [24-26], these frequency zones are known as zones of "negative group velocity". In fact, the phase velocity is negative.

$$w_0 = F^\circ{}_1, \qquad W = F^\circ{}_2, \qquad M = F^\circ{}_3, \qquad Q = F^\circ{}_4 \tag{7.2}$$

In this calculation we take the first six terms (j = 1, 2, ..., 6) in summation (5.1).

We shall use the following notation:

$$\frac{\partial \ldots}{\partial s} = (\ldots)', \qquad \frac{\partial \ldots}{\partial \Omega} = (\ldots)^{\cdot} \tag{7.3}$$

At the saddle points C_{jn} (n = 1, 2, 3, ..., N_j) of the jth integrand function we have

$$s = s_{cjn}, \qquad \varphi'_{cjn} = 1 + \frac{\xi}{\tau}\lambda'_{cjn} = 0, \qquad \frac{\xi}{\tau} = \psi_j \ (m_{cjn})^{-1} \tag{7.4}$$

We calculate the contribution of the saddle points which are on the imaginary axis and which determine nonexponentially decaying waves. At these saddle points, appearing in pairs + C_{jn} and − C_{jn}, we have, on the basis of (6.1)

$$s_{+cjn} = -s_{-cjn} = i\Omega_{cjn}, \qquad \lambda_{+cjn} = -\lambda_{-cjn} = -im_{cjn} \tag{7.5}$$

where Ω_{cjn} and m_{cjn} are real positive quantities. Approximate values of Ω_{cjn} may be found from a plot like Fig. 1, from which also follows how the number N_j of saddle points depends on ψ and j. Further, it is easy to find approximate values of m_{cjn} from known plots of modes [24,25].

For purposes of comparison (see Sections 10-14), Table 1 lists data from the two-dimensional theories. In the Timoshenko-type theory, the shear coefficient if k_T = 0.860.

TABLE 1.

$\mu = 0.3, \ s = i\Omega$

Ω	Three-dimensional theory		Timoshenko-type theory	Kirchhoff theory
	B_{11}	B_{12}	B_{12}	B_{11}
0.0	∞	∞	∞	∞
0.5	−1.9288	−1.8840	−1.9145	−1.9516
1.0	−0.4483	−0.4370	−0.4536	−0.4880
1.5	−0.1761	−0.1741	−0.1864	−0.2169
2.0	−0.08400	−0.08540	−0.09557	−0.1220
2.5	−0.04398	−0.04652	−0.05544	−0.07806
3.0	−0.02416	−0.02688	−0.03488	−0.05423
3.5	−0.01355	−0.01602	−0.02328	−0.03982
4.0	−0.007605	−0.009639	−0.01626	−0.03050
5.0	−0.002197	−0.003293	−0.008788	−0.01952
6.0	−0.0004052	−0.0007435	−0.005252	−0.01355

TABLE 2.

$\mu = 0.3,\ s = i\Omega$

Ω	j	$1000B_{j1}$	$1000B_{j2}$	$1000B_{j3}$	$1000B_{j4}$
7.0	1	+ 0.1175	+0.2721	+ 0.2232	+ 1.763
	2	—14.82	—3.554	—14.72	—26.91
	3	+15.82	+3.010	—88.75	+37.22
	4	+ 7.395	+0.1707	—28.96	+ 3.851
8.0	1	+ 0.2108	+0.6321	+ 0.4545	+ 4.685
	2	— 9.614	—2.791	— 9.482	—23.52
	3	+12.263	+0.8890	—43.79	+11.23
	4	+ 1.590	+1.196	—64.02	+22.13
9.0	1	+0.1803	+0.7139	+ 0.4372	+ 5.956
	2	—6.595	—2.185	— 6.579	—20.400
	3	+7.671	—0.04569	—14.42	— 0.5743
	4	—2.253	+1.648	—78.67	+29.595

TABLE 3.

ψ	j	n	χ	D_1	D_2	D_3	D_4
0.5	1	1	—0.0674	—25.89	—25.77	—1.827	—0.5071
	2	1	1.512	0.02784	0.04221	—0.1230	0.4597
	3	1	4.021	0.02324	0.00348	—0.04936	0.04567
	4	1	5.682	0.00493	— 0.00001	—0.00857	—0.00038
	5	1	7.235	— 0.00018	0.00001	0.00012	0.00033
	6	1	9.676	— 0.00000	0.00005	—0.00045	0.00157
0.6	1	1	—0.0994	—15.21	—15.09	—1.644	—0.5709
	2	1	1.484	0.03611	0.05590	—0.1681	0.5096
	3	1	3.827	0.02726	0.00439	—0.06442	0.05522
	4	1	5.584	0.00733	— 0.00001	—0.01416	—0.00042
	5	1	7.003	0.00028	— 0.00002	—0.00024	—0.00045
	6	1	9.247	0.00000	0.00004	—0.00015	0.00116
0.7	1	1	—0.1400	— 9.338	— 9.234	—1.493	—0.6407
	2	1	1.450	0.04455	0.07106	—0.2263	0.5589
		2	2.244	— 0.1400	— 0.00166	—0.2658	—0.0105
		3	2.240	— 0.1336	— 0.01060	—0.1885	—0.0672
	3	1	3.617	0.03127	0.00541	—0.08235	0.06576
	4	1	5.453	0.01156	0.00002	—0.02760	0.00047
	5	1 *	6.745	0.00102	— 0.00005	—0.00111	—0.00147
	6	1 *	8.781	0.00000	— 0.00001	0.00003	0.00043
0.8	1	1	—0.1911	— 5.813	— 5.722	—1.361	—0.7252
	2	1	1.339	0.0526	0.0876	—0.2896	0.6085
		2	1.955	— 0.0842	0.0082	—0.2427	0.0516
		3	1.860	— 0.0679	— 0.0120	—0.0717	—0.0822
	3	1	3.393	0.0354	0.0065	—0.1044	0.0778
		2	3.948	0.0212	0.0018	—0.0806	0.0223
		3	3.696	0.0113	— 0.0008	—0.0106	—0.0099
	4	1	5.231	0.0048	0.0021	—0.1193	0.0395
		2	5.969	— 0.0086	0.0007	—0.0482	0.0139
		3	5.589	— 0.0041	— 0.0002	—0.0038	0.0033
	5	1	6.459	0.0021	— 0.0001	—0.0030	—0.0028
		2 *	8.011	0.0051	0.0004	—0.0353	0.0101
	6	1 *	8.202	0.0000	0.0000	—0.0002	—0.0006

Table 3 *contd.*:

1	2	3	4	5	6	7	8
0.9	1	1	—0.2576	— 3.529	— 3.453	—1.243	—0.8521
	2	1	1.359	0.0591	0.1054	—0.3750	0.6609
		2	1.704	— 0.0773	0.0158	—0.2963	0.0979
		3	1.163	— 0.0499	— 0.0138	—0.0492	—0.1130
	3	1	3.153	0.0401	0.0078	—0.1330	0.0923
		2 *	3.477	0.0228	0.0029	—0.1076	0.0371
	4	1	4.872	— 0.0003	0.0025	—0.1207	0.0445
		2 *	5.294	— 0.0087	0.0012	—0.0688	0.0222
	5	1	6.144	0.0039	— 0.0002	—0.0072	—0.0044
		2 *	7.139	0.0056	0.0006	—0.0528	0.0157
	6	1 *	7.751	0.0000	— 0.0001	—0.0016	—0.0020
1.0	1	1	—0.3550	— 2.007	— 1.956	—1.309	—1.452
		2	—0.3505	— 0.6361	— 0.6448	—0.6009	—1.069
	2	1	1.299	0.0619	0.1248	—0.4925	0.7210
		2	1.478	— 0.0744	0.0270	—0.3848	0.1652
	3	1	2.896	0.0459	0.0096	—0.1747	0.1121
		2	3.043	0.0278	0.0047	—0.1489	0.0584
	4	1	4.473	— 0.0026	0.0031	—0.1449	0.0547
		2	4.665	— 0.0093	0.0018	—0.0994	0.0338
	5	1	5.793	0.0074	— 0.0003	—0.0185	—0.0063
		2	6.321	0.0067	0.0009	—0.0782	0.0230
	6	1	7.189	0.0001	— 0.0001	—0.0056	—0.0036
		2	7.998	— 0.0036	0.0005	—0.0646	0.0165
1.1	2	1	1.223	0.0547	0.1479	—0.6885	0.8087
		2	1.278	— 0.0712	0.0512	—0.5605	0.3037
	3	1	2.616	0.0581	0.0129	—0.2615	0.1513
		2	2.644	0.0419	0.0086	—0.2388	0.1058
	4	1	4.038	— 0.0058	0.0041	—0.1945	0.0730
		2	4.077	— 0.0109	0.0031	—0.1633	0.0576
	5	1	5.388	0.0206	— 0.0001	—0.0903	—0.0033
		2	5.549	0.0092	0.0014	—0.1220	0.0232
	6	1	6.593	0.0003	— 0.0002	—0.0162	—0.0051
		2	7.043	— 0.0037	0.0007	—0.0906	0.0213
1.2	6	1	5.954	0.00053	—0.00016	—0.0457	—0.0047
		2	6.131	— 0.00363	0.00082	—0.1260	0.0261

8. By the usual methods [28, 29] one may obtain formulas of the following form for the unknown quantities (7.2):

$$F_i^\circ \approx \frac{A_0}{\sqrt{\pi\psi\tau}} \sum_{j=1}^{\infty} \sum_{n=1}^{N_j} f_{jni} \qquad (\tau \to \infty) \quad (i = 1, 2, 3, 4) \tag{8.1}$$

To simplify the writing of the formulas for the functions f_{jni}, we shall drop the indices j and n.

In the case $m_c^{\cdot\cdot} \neq 0$, a known computational scheme [28] gives the following formulas for the first approximation:

$$f_k = D_k T, \qquad f_4 = D_4 T^*, \qquad D_i = B_{+ci}\,|m_c^{\cdot\cdot}|^{-1/2} \quad (k = 1, 2, 3;\ i = 1, 2, 3, 4) \quad (8.2)$$

where

$$T = \pm \cos \tau\chi - \sin \tau\chi, \qquad T^* = \cos \tau\chi \pm \sin \tau\chi, \qquad \chi = \Omega_c - \psi m_c \quad (8.3)$$

Here the upper signs are used for even n (for the case $m_c^{\cdot\cdot} < 0$) and the lower signs are used for odd n (for the case $m_c^{\cdot\cdot} > 0$). The functions B_{+ci} are evaluated from (6.6) at the saddle points " $+C$ " on the positive semi-axis $s = i\Omega$.

For brevity's sake we shall not introduce the more accurate formulas for the saddle-point method which were obtained on the basis of [28]. With the help of the approximate formulas of [24], these were used in the analysis of the accuracy of (8.2) for $\psi \to 0$. It was established that for $j \geqslant 2$, formulas (8.2) are in error to the order of τ^{-1}. Starting with (6.3), one may arrive at a similar conclusion with respect to the first mode ($j = 1$). However, in the case of $j = 1$, the standard formulas (8.2) have an essential error of another form that is associated with the effect of the singular point $s = 0$. This insufficiency of the saddle-point method may be removed by using a procedure for calculating the effect of the singularity that was described in [29]. A corresponding analysis showed that in calculating quantities (7.2), one should take into account corrections to the components w_0, W and M. In the calculation of these corrections for the case $\psi \to 0$, the contribution of all of the modes is determined by the formulas for the first approximation to an accuracy of τ^{-1}, and, hence, they can be practicably applied for comparatively small τ. Generally speaking, the error of these formulas increases with increasing ψ. The case of $|m_c^{\cdot\cdot}| \ll 1$ turns out to be particularly "dangerous".

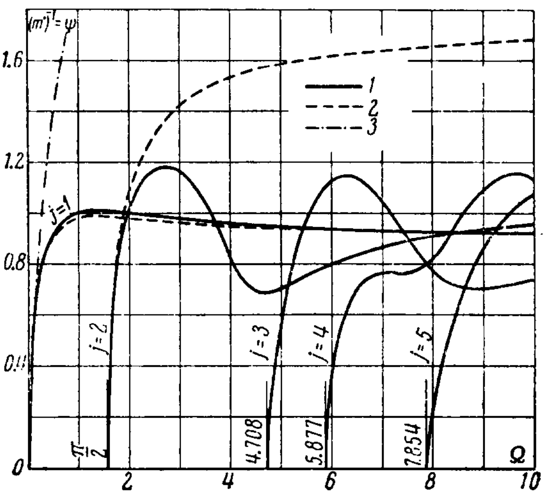

Fig. 1.

In the main, formulas (8.2) become unsuitable for $m_c^{\cdot\cdot} = 0$. For

$|m_c''| \ll 1$ they give acceptable results only for very large τ. Hence, in this case we apply the well-known method [29] of reducing the problem to the calculation of the Airy function. A detailed analysis of the contour of integration gives the following computational formulas:

$$f_k = K_k\,(Y \sin \tau\chi \pm X \cos \tau\chi) \qquad (k = 1,\, 2,\, 3) \tag{8.4}$$

$$f_4 = K_4(\pm X \sin \tau\chi - Y \cos \tau\chi)$$

$$K_i = (4\tau\psi)^{1/4}\,|\,m_c'''\,|^{-1/3} B_{+ci} \qquad (i = 1,\, 2,\, 3,\, 4) \tag{8.5}$$

As in formulas (8.3), it is necessary to use the upper signs for even n in (8.4), and the lower signs for odd n. For the sake of brevity, we shall not introduce the formulas for X and Y; these are easily obtained by means of [29]. We remark that for $m_c'' \neq 0$ and $\tau \to \infty$, formulas (8.4) and (8.5) go over into (8.2). In the particular case of $m_c'' = 0$, two saddle points merge and we have the so-called Airy phase [14,20].

9. The computations were carried out with the help of the M-3 type electronic conputer. An abbreviated summary of the results of computations according to (8.2) is presented in Table 3.

In Table 3, an asterisk indicates values of ψ and j for which there exist still other saddle points that yield data that are not tabulated. At these points the wave groups are commensurate in frequency with groups of uncomputed modes $j \geqslant 7$ and are of insignificant amplitude.

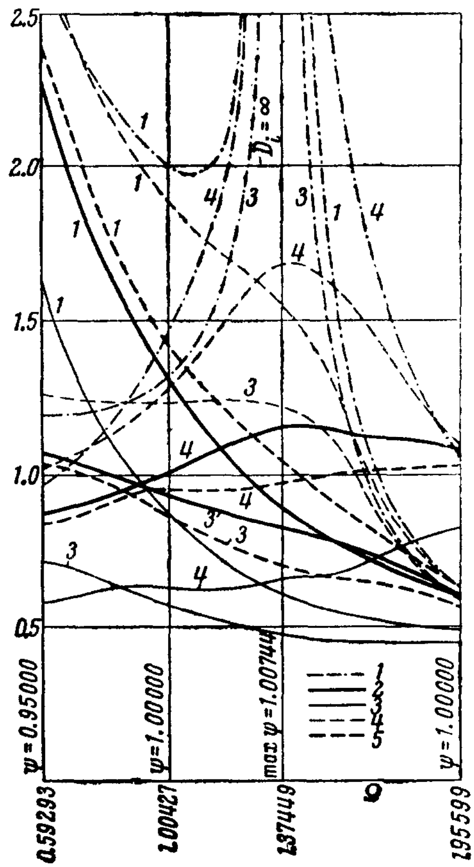

Fig. 2.

The first mode has an Airy phase where $\Omega = 1.37449$ and $\psi = 1.00744$, and the second mode has two Airy phases, where $\Omega = 2.71053$, $\psi = 1.18313$, $\Omega = 4.69706$ and $\psi = 0.690622$, etc. Formulas (8.4) were used in the neighborhood of these points (Fig. 1). Likewise, the problem of the transition from formula (8.2) to (8.4) was investigated. It was established that for comparatively small τ these formulas give essentially different results over a rather wide range of frequencies Ω, but that with increasing τ this range rapidly decreases. For a numerical illustration of this fact, in Fig. 2 are compared the amplitudes D_i in formula (8.2) and XK_i and YK_i in formulas (8.4) in the Airy phase region for the first mode; $i = 1$ refers to the normal displacement w_0, $i = 3$ to the moment M and $i = 4$ to the transverse force Q. Curve 1 represents $(-D_i)$, Curves 2 and 3, respectively, represent $(-XK_i)$ and YK_i for $\tau = 10$, Curves 4 and 5, respectively, represent $(-XK_i)$ and YK_i for $\tau = 100$. It is seen that the amplitudes already differ insignificantly for $\tau = 100$ and $\Omega < 0.6$ and $\Omega > 1.9$.

TABLE 4.

	$j=2$	$j=3$	$j=4$
w_0	0.11	0.09	0.02
W	0.16	0.01	10^{-4}
M	6.7	2.7	0.47
Q	91	9.0	0.08

TABLE 5.

ψ	j	Kirchhoff theory			Timoshenko-type theory		
		D_1	D_3	D_4	D_1	D_3	D_4
0.5	1	—30.11	1.978	0.5061	—25.76	—1.830	—0.5088
	2	—	—	—	0.03735	—0.1129	0.4513
0.6	1	—19.08	1.804	0.5544	—15.08	—1.648	—0.5741
	2	—	—	—	0.04921	—0.1528	0.4986
0.7	1	—12.98	1.670	0.5989	—9.216	—1.499	—0.6469
	2	—	—	—	0.06207	—0.1997	0.5439
0.8	1	—9.298	1.562	0.6402	— 5.702	—1.372	—0.7381
	2	—	—	—	0.07574	—0.2549	0.5881
0.9	1	—6.926	1.473	0.6791	— 3.424	—1.265	—0.8858
	2	—	—	—	0.08991	—0.3204	0.6314

10. *Case of* $\psi < 1/2$. In the w_0, W and M portion of the quantities the contribution of the first mode dominates; in the Q portion the first and second modes dominate. The contribution of the other modes rapidly decreases with an increase in the order number j. The maximum amplitudes of the modes $j = 2$, 3 and 4 as compared with the amplitude of the mode $j = 1$ are indicated in Table 4, where all quantities are given as percentages.

By substituting expansions (6.3) and (6.5) into (5.1), one may find the contribution of the first mode for $\psi \to 0$. This method yields contour integrals for the Kirchhoff theory for which inverses are known exactly [1]. For $\psi < 1/2$ the maximum error in such an approximation of the contribution of the first mode is 12 per cent for w_0, 11 per cent for M and 0.2 per cent for Q. In the w_0 and M portion the contribution of the first mode prevails, hence these quantities may be determined by the Kirchhoff theory with more or less acceptable accuracy. This cannot be said with respect to Q.

The Timoshenko-type theory [5,9,14] approximates the contribution of the first and second modes for the quantities W, M and Q. In the first mode portion the error in the approximation does not exceed 0.5 per cent. The contribution of the second mode is much more poorly approximated, but the total error of the approximation of the contribution of the first two modes does not exceed 2 per cent. For practical purposes, the Timoshenko-type theory for $\psi < 1/2$ guarantees roughly the same accuracy as the first two modes of three-dimensional theory.

If the contribution of the first mode in the w_0, W and M portion is determined by taking into account the correction due to the singular point $s = 0$ (see Section 8), then the formulas for the first approximation in the saddle-point method (8.2) allow one to compute the contour integrals to within an error of order τ^{-1}.

11. *Case of* $1/2 \leqslant \psi \leqslant \psi_R = 0.9274$. From the numerical results of Table 3 and Fig. 2 it follows that in the w_0 and W portion the domination of first mode values is retained. In the M portion, the role of the modes $j \geqslant 2$ increases rapidly. For example, for $\psi = 0.9$ the amplitude of mode $j = 2$ is about 30 per cent of the first mode amplitude; that of modes $j = 3$, 4 about 11 per cent, and of mode $j = 5$ about 4 per cent. With increasing ψ, the significance of modes $j \geqslant 3$ gradually increases in the Q portion as well, but much more slowly than in the M portion. If four modes are calculated, the maximum amplitudes of the discarded modes are about 5 per cent of the first mode amplitudes in the M portion, and about 2 per cent in the Q portion.

One of the characteristic properties in the range being considered is the existence of Airy phases in modes $j \geqslant 2$. For $\psi = \psi_{aj}$, corresponding to an Airy phase of mode j, the oscillations of the mode decrease more slowly than for other values of ψ by a factor of $\tau^{1/6}$. Hence, for very large τ values, the Airy phase should become visible. However, in an estimate of the relative roles of the modes this fact is not of essential significance; the case of $\tau > 1000$ is hardly of practical interest, if one is to apply the computations for the semi-infinite plate model to the engineering design of structural elements that have finite length.

Table 5 gives an estimate of the accuracy of the two-dimensional theories. In the construction of the table, a shear coefficient of $k_T = 0.860$ [5,9,14] was used in the Timoshenko-type theory (following a suggestion of Mindlin [30]).

From a comparison of the data in Tables 3 and 5, it follows that the error in the Kirchhoff theory rapidly increases with increasing ψ. For $\psi = 0.9$, the difference in the w_0 portion is already roughly two-fold; in the M, Q portion it is likewise very large. This of course could have been anticipated from Table 1.

As a consequence of the dominant role of the first mode, the W portion is determined rather well by the Timoshenko theory. In the M, Q portion, the approximations become rather coarse for $\psi > 0.7$, even though the contributions of the saddle points $j = 1$, $n = 1$ and $j = 2$, $n = 1$ are determined with small error.

This is associated with the essential contribution (for $\psi > 0.7$) of other saddle points (especially in the M portion) in the $j = 2$ mode, and likewise saddle points in $j = 3, 4$ modes.

12. *Case of* $\psi_R < \psi \leqslant 1.00744$. The upper limit of this range is set by the Airy phase of the first mode (maximum value of $\psi_1(\Omega)$). The relative roles of the modes, compared with the Airy phase of the first mode, is here essentially dependent on τ. If we limit ourselves to $10 < \tau < 1000$, then in the w_0, W portion the contribution of the first mode is 5 to 10 times greater than that of the second mode, while in the M, Q portion the first and second modes have roughly the same value. The contribution of the subsequent modes rises significantly. For an accurate determination of M and Q it is necessary to compute 4 or 5 modes.

The Kirchhoff theory is inapplicable. The error in the Timoshenko-type theory increases not only because of an increase in the contribution of modes $j \geqslant 3$, but also because of an essential increase in the error of approximation of modes $j = 1, 2$. Indeed, in this case the contributions of the saddle points $n = 1$ and $n = 2$ of the seond mode are almost the same, but the Timoshenko-type theory approximates only the contribution of the points $n = 1$. The contribution of the points $n = 2$ of the first mode is completely incorrectly determined for $\Omega > 4$ (this could have been predicted from Table 1). The curve $\psi_1(\Omega)$ deviates from the correct curve in the region of its maximum (Airy phase). From what has been indicated, the Timoshenko-type theory gives inaccurate results for M and Q, while for W it gives more or less acceptable results only for $0.95 \leqslant \psi \leqslant 0.98$.

Note. Miklowitz called attention to this phenomenon in [15]. He suggested that a narrow range of values of ψ around ψ_R be excluded from consideration (in the applications of the Timoshenko-type theory). From the present paper, it follows that this method can have some significance only in the W portion.

13. *Case of* $\psi > 1.00744$. The contribution of the first mode is absent if one does not compute the exponentially decaying waves that are associated with saddle points off the imaginary axis. The contribution of subsequent modes decreases more slowly with increasing j than in the region of ψ considered previously. Up to cut-off in the second mode ($\psi \leq 1.18313$) it is possible to obtain more or less accurate results by taking 5 or 6 modes into account. For larger ψ values, one must take into account an increasing number of higher modes, which makes the mathematical apparatus that has been applied ineffective for practical use. The larger ψ is, the more the transverse section of the plate deforms; but the quantities w_0, W rapidly decrease with increasing ψ. The two-dimensional Kirchhoff and Timoshenko-type theories are inapplicable.

14. The conclusions presented on applicability were based on results obtained by the saddle-point method for $\tau \gg 1$. However, starting with the data in Tables 1 and 2, and taking (5.4) into account, it is not difficult to conclude that at the beginning of the motion the role of modes $j > 2$ increases and the corresponding accuracy of the Timoshenko-type theory decreases more rapidly than it increases.

On the other hand, it should be noted that in the study of wave motion in real structures, the Timoshenko-type theory may give better results than in the example considered. In its favor is the fact that the loads that act have a smoother variation with time; likewise, damping occurs because of energy absorption.This decreases the relative role of high-frequency wave groups.

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